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Some aspects of the harmonic balance method applied to the clarinet

Claudia Fritz [a,*], Snorre Farner [b], Jean Kergomard [c]

[a] IRCAM, 1 pl. Igor Stravinsky, 75004 Paris, France

[b] Department of Electronics and Telecommunications, NTNU, O.S. Bragstads plass 2B, 7491 Trondheim, Norway

[c] Laboratoire de Mecanique et d'Acoustique, CNRS, 31 Chemin Joseph Aiguier, 13402 Marseille, France

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Abstract

The clarinet has been extensively studied by various theoretical and experimental techniques. In this paper, the harmonic balance method (HBM), a numerical method mainly working in the frequency domain, has been applied to solve a simple nonlinear clarinet model consisting of a linear exciter (for the reed) nonlinearly coupled to a linear resonator with visco-thermal losses (for the pipe). A recent and improved implementation of the HBM for self-sustained instruments has allowed us to study the model theoretically when including dispersion in the pipe or mass and damping terms in the reed model. The resulting periodic solutions for the internal pressure spectrum and the corresponding playing frequency are shown to align well with previous theoretical and experimental knowledge of the clarinet. Finally, we present and briefly discuss a few (probably unstable) oscillation regimes both with the HBM and with a real clarinet.

Keywords: Clarinet; Harmonic balance; Internal pressure spectrum; Playing frequency; Oscillation regimes

* Corresponding author. Tel.: +33 1 4478 4843; fax: +33 1 4478 1540.
E-mail address: fritz@ircam.fr (C. Fritz).

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1. Introduction

After the first theoretical attempt to derive the spectrum of reed instruments by Worman [1], improvements have been made to determine the nature of the bifurcation and the spectrum at small oscillations [2], the influence of the main control parameters on the square signal [3], and the transition between small oscillations and the square signal [4]. A numerical method called the harmonic balance method (HBM) has been adopted and developed by Schumacher [5] and Gilbert et al. [6] for self-sustained instruments, and an approximate analytical method called the variable truncation method (VTM) was established by Kergomard et al. [4] to obtain analytical results, which were compared with the results of the HBM.

A good reason for employing frequency-domain methods like HBM and VTM is that the solutions found do not depend on the history as they do with time-domain methods. All solutions, stable and unstable, can, in principle, thus be found, which is convenient when studying the influence and control of the different parameters of a given model (cf. [7]). As a natural consequence, HBM and VTM are not useful for studies of transients.

An efficient computing tool using the HBM has been developed by Farner (see [7]) enabling us to present solutions not presented earlier. It makes it possible to follow a solution as a parameter changes, for example from small to large oscillations, and thereby easily study the influence of the parameters such as mouth pressure, viscothermal dissipation in the pipe, and dispersion, as well as the effective stiffness, mass, and damping of the reed.

In the next two sections, we briefly describe a common physical model for the clarinet and the two methods, HBM and VTM, for solving the model equations. Then we study aspects of this model by starting with a simplified version of it and successively adding the effects of dispersion (inharmonicity of the resonator) and the influence of the reed resonance (reed mass and damping) to end up with the described model.

We mainly restrict the study to the first harmonic of the pressure in the mouthpiece and note that exact numerical solutions would be obtained by the HBM if infinitely many harmonics were taken into account.

2. Model of the clarinet

The clarinet may be modelled as a self-sustained oscillator with a linear exciter (the reed) [8] that is coupled nonlinearly to a linear resonator (the pipe). A common model is described in this section where the simplifications ignore, for instance, recent knowledge on torsional modes of the reed [9–11] and the interaction between the reed and the mouthpiece lay [12] as well as nonlinear effects in the pipe [13,14]. Furthermore, any acoustic effects of the player's vocal tract are omitted. Despite its simplicity, this model incorporates many important characteristics of the real clarinet [15,9]. A sketch of the mouthpiece is shown in Fig. 1 including the meaning of the physical quantities used. Dimensionless quantities are introduced (marked

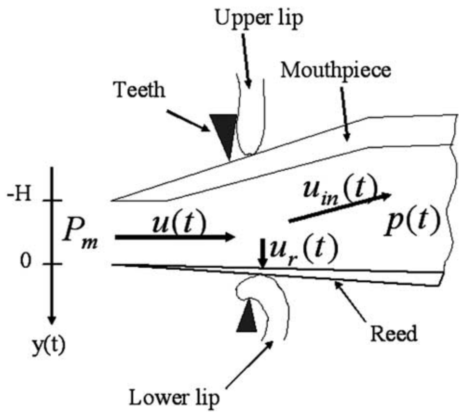

Fig. 1. Schematic view, not to scale, of the clarinet mouthpiece with physical quantities.

with a tilde in this section only) to generalize the graphs and facilitate the later developments in the VTM.

2.1. The reed

The exciter of a clarinet is the reed, which converts energy supplied by the flow of air from the mouth (at elevated pressure p_m, assumed constant) into acoustic energy. Following Wilson and Beavers [16], we treat the reed as a linear spring with mass per unit area μ_r, resonant frequency ω_r, and specific damping g_r. Its displacement y from the equilibrium position is then related to the pressure p in the mouthpiece by:

$$\ddot{y}(t) + g_r\dot{y}(t) + \omega_r^2 y(t) = \frac{1}{\mu_r}(p(t) - p_m), \tag{1}$$

where the dots signify time derivatives. The bore of the instrument is represented by its fundamental resonance with angular frequency ω_p and a series of higher resonances (cf Section 2.2). The maximum negative value of y is $-H$, at which the reed closes, when the mouth pressure is equal to a certain value p_M. Using a tilde to indicate dimensionless quantities, we write:

$$\begin{aligned} \tilde{y} &= y/H && \text{reed position} \\ \tilde{P} &= p/p_M && \text{acoustic pressure in the mouthpiece} \\ \tilde{t} &= t\omega_p && \text{time} \end{aligned} \tag{2}$$

We nondimensionalize the acoustic pressure in the mouth, the blowing pressure, in terms of p_M:

$$\gamma = p_m/p_M \tag{3}$$

The dimensionless version of Eq. (1) is then:

$$M\ddot{\tilde{y}}(\tilde{t}) + R\dot{\tilde{y}}(\tilde{t}) + K\tilde{y}(\tilde{t}) = \tilde{p}(\tilde{t}) - \gamma, \tag{4}$$

where K is the dimensionless stiffness, M the dimensionless mass and R the dimensionless damping [7]. We have $K = 1$ because the reed closes ($y = -H$) when $p_m = p_M$, thus $M = (\omega_p/\omega_r)^2$ and $R = \omega_p g_r/\omega_r^2$.

Note that in this system the player's embouchure may be incorporated to some degree, in that different positions of, and forces applied by, the lips yield different values of M and R.

A simplified description can be made by setting $M = R = 0$:

$$\tilde{y}(\tilde{t}) = \tilde{p}(\tilde{t}) - \gamma. \tag{5}$$

This simple spring representation of the reed may be good if the playing frequency is low compared to the reed frequency ($\omega_p \ll \omega_r$) and the harmonics around ω_r so small that they do not interact with the resonance peak of the reed. The resonance frequency of the reed, $\omega_r/2\pi$, is normally above 2000 Hz.

2.2. The pipe

The pipe is usually characterized by its input impedance [17], which describes its resonances. We are interested in the oscillation mechanism, and high frequencies are relatively unimportant to our study, so we make some severe approximations. We assume at first that the impedance maxima of the bore are exactly harmonic, and that their relative heights are determined only by visco-thermal losses. In practice, the frequencies of the first few impedance maxima of a real clarinet are in approximately harmonic ratios, while the higher frequency peaks are successively further from harmonic [18]. Some of the frequency-dependent effects thus neglected here are those due to the shape of the bell, the mouthpiece, the tone holes and the diameter variations along the bore. Other frequency-dependent effects are those of the radiation impedance at the end (incorporated to first order as a length correction) and the wave dispersion (discussed in Section 5).

The oscillation may be regarded as the interaction between the standing wave in the bore (produced by reflections at both ends) and the reed, where the volume velocity wave $u(t)$ interacts with the pressure $p(t)$ in the mouthpiece. In the Fourier domain we write with dimensionless quantities

$$\tilde{P}(\omega) = \tilde{Z}(\omega)\tilde{U}(\omega), \tag{6}$$

where the capital letters P and U are used for the Fourier transforms of the time domain quantities p and u. The dimensionless input impedance is given by [4,17]:

$$\tilde{Z}(\omega) = \frac{Z}{Z_0} = \mathrm{j}\tan(kl) \tag{7}$$

with

$$kl = \frac{\pi f}{2f_0} + (1 - \mathrm{j})\psi\eta\sqrt{\frac{f}{f_0}}, \tag{8}$$

where $f = \omega/2\pi$ being the frequency, ψ being related to the Prandtl number [3] ($\psi \simeq 1.3$ for common conditions in air), $f_0 = c/4l$, l being the length of the pipe, and η is a dimensionless loss parameter (see [3]).

In Eq. (8), the real part of the last term on the right is the dispersion term due to visco-thermal effects (i.e., frequency depending on the sound velocity), while the imaginary part is due to visco-thermal losses.

If we ignore the dispersion term in addition to the aforesaid approximations, the resonance frequencies are harmonically related, and the input impedance for the nth harmonic can be written with the following simplified formula taking advantage of the fact that η generally is small (typically 0.02 for the clarinet):

$$Z_n = \begin{cases} 1/(\sqrt{n}\psi\eta), & \text{for odd } n, \\ \sqrt{n}\psi\eta, & \text{for even } n. \end{cases} \tag{9}$$

2.3. The nonlinear coupling

When there is a pressure difference $\tilde{p} - \gamma$ between inside the mouth and inside the mouthpiece, air will flow with a volume flow $\tilde{u} = uZ_0/p_M$, where $Z_0 = \rho c/S$ is the characteristic input impedance of the pipe with air density ρ, sound speed c in air, and cross section S of the pipe. Assuming some hypotheses and in particular that the system is sufficiently stationary for Bernoulli's law to be valid [19], a nonlinear expression with dimensionless quantities [3] can be obtained for the volume flow:

$$\tilde{u}(\tilde{p}, \tilde{y}) = \zeta(1 + \tilde{y})\sqrt{|\gamma - \tilde{p}|}\,\text{sign}(\gamma - \tilde{p}) \tag{10}$$

for $\tilde{y} > -1$, and otherwise 0, signifying in simple terms that the reed bends up and blocks the opening of the mouthpiece for a part of the oscillation period, though the real behavior is slightly different [12]. This last case is more complex so we limit the study to the non-beating reed regimes and stop the curves before the reed starts to beat.

The "embouchure" parameter

$$\zeta = Z_0 wH\sqrt{\frac{2}{\rho p_M}} \tag{11}$$

characterizes the mouthpiece and the mouth position of the musician, w being the width of the reed channel. ζ depends on the stiffness of the reed, on the lips' position, and on the ratio between the cross sections of the reed canal and of the pipe. Its value is important especially for attack transients.

Strictly speaking, there is also a contribution to the volume flow from the reed displacement to the volume flow of air (u_r in Fig. 1). This is often included in the impedance of the pipe by a length correction [20], which does not change the harmonicity of the pipe's resonances.

If a simple reed model is used, i.e., Eq. (5), then Eq. (10) simplifies to

$$\tilde{u}(\tilde{p}) = \zeta(1 + \tilde{p} - \gamma)\sqrt{|\gamma - \tilde{p}|}\,\text{sign}(\gamma - \tilde{p}) \tag{12}$$

for $\tilde{p} > \gamma - 1$ and 0 otherwise.

This gives us a set of three Eqs. (4), (6) and (10) to be solved. As noted earlier, only dimensionless quantities will be used in the following, so we omit the tilde. One of our purposes is to calculate the playing frequency, which depends on the first resonance of the pipe and the effective resonance of the reed, so it is convenient to introduce these two constants: $f_p = \omega_p/2\pi$ and $f_r = \omega_r/2\pi$. Note that f_p/f_0 if we disregard dispersion.

3. Solving methods

3.1. The harmonic balance method

The nonlinear problem can be solved by means of the harmonic balance method (HBM), which is a numerical method to calculate the steady-state spectrum of periodic solutions of nonlinear oscillating systems. The method can be used on free-oscillating systems [21] and extended to self-sustained musical instruments such as the clarinet [6].

We assume that the Fourier series of the pressure in the mouthpiece may be truncated to N_p harmonics (partials) plus the DC component. Separating real and imaginary parts, we represent the pressure spectrum by $\vec{P}$ with $2N_p + 2$ real components.

The three model Eqs. (4), (6) and (10) may be formulated by the fixed point representation $\vec{F}(\vec{P}, f)$ where $\vec{P}$ must satisfy

$$\vec{P} = \vec{F}(\vec{P}, f) \tag{13}$$

for a playing frequency f, which is unknown since the interaction with the reed and dispersion will cause f to differ from the frequency of the first resonance of the resonator. This gives $2N_p + 2$ equations and $2N_p + 3$ unknowns.

A periodic signal is invariant to a shift in the time domain, so the solution can be shifted to make the first harmonic real and thus its imaginary part zero. This reduces the number of unknowns to $2N_P + 2$, and a finite number of solutions of Eq. (13) may then be found by searching for a root of

$$\vec{G}(\vec{P}, f) = \frac{\vec{P} - \vec{F}(\vec{P}, f)}{P_1}, \tag{14}$$

i.e., $\vec{G} = 0$, except for the component corresponding to the imaginary part of the first harmonic. The nonzero denominator P_1 prevents the trivial solution $\vec{P} = 0$. From an estimated solution $(\vec{P}^i, f^i)$ we use the Newton–Raphson method, which follows the locally steepest descent direction of $\vec{G}(\vec{P}, f)$ and returns a step $(\Delta\vec{P}, \Delta f)$ to a point $(\vec{P}^{i+1}, f^{i+1})$ normally closer to a solution. Note that the change in the playing frequency is treated directly in the Newton–Raphson step of the method, as detailed by Farner et al. [7].

The iteration process may be stabilized by, for example, a backtracking routine [22], which avoids a diverging step caused by a locally unfavorable shape of $\vec{G}$.

We expect that a solution of Eq. (13) would be a periodic solution of our Eqs. (4), (6) and (10) except for the aliasing problem, which is diminished by a sufficiently high sampling rate. It must be noted that the method says nothing about the stability of the solution.

The method is described in detail in [7], which also presents a realization of the HBM for self-sustained musical instruments through a free computer program called *Harmbal* [23]. The program is made to handle more general models composed of a linear exciter and resonator with a nonlinear coupling. This program was used for the present calculations in conjunction with a Perl script (called *hbmap*) for continuation, i.e., using one result to calculate the next when varying a parameter.

3.2. The variable truncation method

The variable truncation method (VTM) [4] is an analytical method based on truncation of the Fourier series obtained from the governing equations and separation of the symmetric and antisymmetric harmonics. In the present system, we simplify the nonlinear equation to Eq. (12) by ignoring the mass and damping of the reed, then we expand it to a third-order polynomial in p, which is sufficient for small oscillations, i.e., close to the oscillation threshold:

$$u(p) = u_{00} + Ap + Bp^2 + Cp^3, \tag{15}$$

where

$$u_{00} = \zeta(1-\gamma)\sqrt{\gamma}, \quad A = \zeta\frac{3\gamma-1}{2\sqrt{\gamma}}, \quad B = -\zeta\frac{3\gamma+1}{8\gamma^{3/2}}, \quad C = -\zeta\frac{\gamma+1}{16\gamma^{5/2}}. \tag{16}$$

The internal pressure $p(t)$ is then written as a Fourier series with harmonics P_i like in the HBM, and assumed to contain only odd harmonics (i = 1,3,5,...) with P_1 real. In brief, the volume flow $u(t)$ is decomposed into a symmetric part ($u_s = u_{00} + Bp^2$) and an antisymmetric one ($u_a = Ap + Cp^3$), and u_a is truncated to the Nth harmonic and combined with Eq. (6). This gives a system of $N + 1$ nonlinear, complex equations and equally many unknowns, including the playing frequency f (as for the HBM). However, as shown by Kergomard et al. [4], the result when truncating the nth equation to the order n is not bad even for a square signal. The VTM thus takes advantage of the fact that higher harmonics have a weak influence on the lower ones.

Close to the threshold, the signal is almost sinusoidal [2] so it is sufficient to truncate the Fourier series to the first harmonic, in which case the VTM reduces to a classical first-harmonic method. Eq. (15) thus becomes

$$U_1 = AP_1 + 3CP_1^3, \tag{17}$$

and by applying the admittance $Y_1 = 1/Z_1$, Eq. (6) becomes $U_1 = Y_1P_1$ for the first harmonic, thus

$$P_1^2 = \frac{Y_1 - A}{3C}. \tag{18}$$

In contrast to the HBM, the first harmonic calculated with the VTM depends only on the degree of expansion, not the number of harmonics. Thus, in the following studies we will only compare the first harmonic of the VTM, even though we consider many harmonics with the HBM.

4. Simple reed model

Although the HBM is a powerful method, we start with the elementary case of a simple reed model without mass and damping, Eq. (5), and with no dispersion, Eq. (9). This enables us to verify the results with the analytical method VTM. Later on we use the HBM on cases where analytical solutions are difficult to find.

4.1. Playing frequency

When dispersion is not taken into account, the resonance frequencies of the pipe are harmonically related. The playing frequency f is thus the same whatever the value of the mouth pressure and however many harmonics are taken into account. From Eq. (18), Y_1 must be real (as P_1 is real), which is satisfied only for f being a resonance frequency of the pipe. Thus for the lowest register, we get $f = f_p = f_0$, where f_0 was defined in Eq. (8).

4.2. The oscillation threshold

Near the oscillation threshold, Eq. (12) can be approximated by the third-order expansion (15) and solved by the one-harmonic VTM: Eq. (18) implies that the oscillation threshold ($P_1 = 0$) is given by $Y_1 = A$. From Eq. (9) it follows that $Y_1 = \psi\eta$, and thus the weakest blowing pressure that gives oscillation is

$$\gamma_t h = \frac{1}{9}\left(\psi\frac{\eta}{\zeta} + \sqrt{3 + \left(\psi\frac{\eta}{\zeta}\right)^2}\right)^2 \simeq \frac{1}{3} + \frac{2}{3^{3/2}}\psi\frac{\eta}{\zeta} \simeq \frac{1}{3} + 0.5\frac{\eta}{\zeta}. \quad (19)$$

This result is compared with the results of the HBM for a few sets of ζ and η in Table 1. The table shows good agreement between the two methods and reveals that the musician must blow harder in the case of higher losses η (γ_{th} increases, but less hard when, for instance, the opening H is decreased ($\zeta \propto \sqrt{H}$) decreases).

4.3. Influence of the number of harmonics

Before using the HBM far from the oscillation threshold, a study of the influence of the number of harmonics N_p is required to optimize the precision while minimizing the calculation time. Close to the threshold the pressure wave is almost sinusoidal [2], and thus the first harmonic P_1 varies little with N_p. Further from the threshold, an increasing number of harmonics become important, and this influences p_1, as

Table 1
Comparison of oscillation thresholds γ_{th} obtained using the VTM, Eq. (19), and the HBM with $N_p = 9$ for some values of ζ and η

	Method	$\eta = 0.01$	$\eta = 0.02$	$\eta = 0.03$
$\zeta = 0.2$	VTM		0.3834	
	HBM		0.3872	
$\zeta = 0.4$	VTM	0.3458	0.3583	0.3708
	HBM	0.3461	0.3593	0.3730
$\zeta = 0.6$	VTM		0.3500	
	HBM		0.3504	

shown in Fig. 2 for N_p up to 49 harmonics. We have chosen to stop at $N_p = 9$ as P_1 varies little, and only at high γ, when we add the 11th harmonic, or even another 40 harmonics.

In other contexts, for instance for less visco-thermal losses, more harmonics may be needed, and it should be made clear that even if higher harmonics are relatively weak, they are important for the perception of the corresponding sound.

Here, and in all following figures, the curves calculated using the HBM end where the reed starts to beat. The fact that the beating threshold varies with N_p is due to large overshoots in $u(t)$ for small N_p as shown in Fig. 3. Beating occurs only for $N_p = 1$ and 3 at $\gamma = 0.494$.

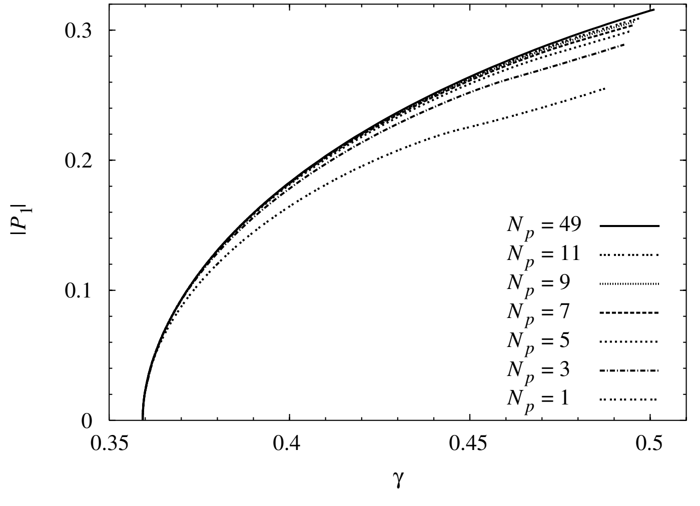

Fig. 2. P_1 versus γ for different values of N_p for $\zeta = 0.4$, $\eta = 0.02$, calculated using the HBM. The curves are cut at the beating threshold.

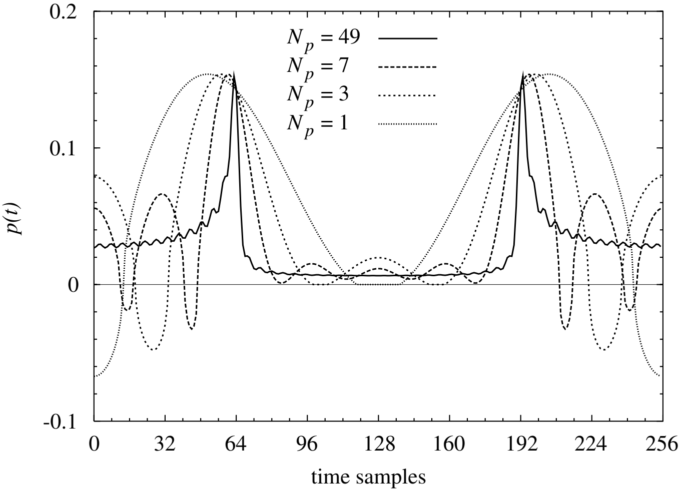

Fig. 3. One period of the oscillation of the volume flow for $\gamma = 0.494$ for various N_p ($\zeta = 0.4$, $\eta = 0.02$).

4.4. Amplitude of first harmonic

Fig. 4 shows the variation of the first harmonic P_1 with respect to the blowing pressure γ obtained using the VTM, i.e., Eq. (18), and using the HBM for $N_p = 9$ harmonics using the cubic expansion and the exact nonlinearity, Eqs. (15) and (10), respectively.

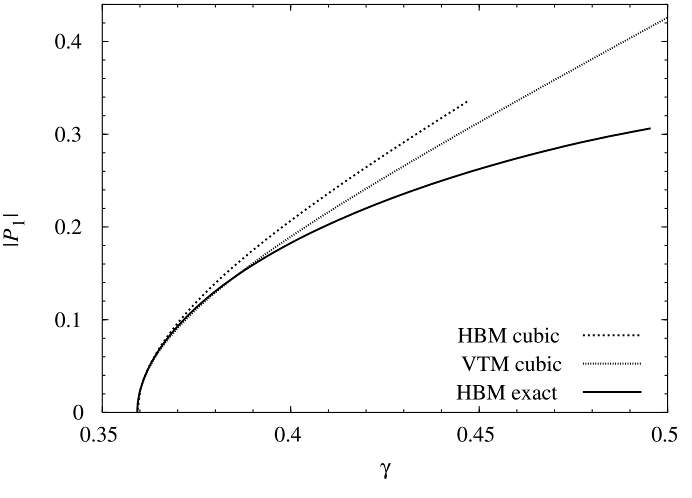

Fig. 4. Comparison of the first harmonic between the VTM cubic (Eq. (18)) and the HBM using the cubic and exact versions of the nonlinear equation, $N_p = 9$, $\zeta = 0.4$, $\eta = 0.02$.

As expected, the first harmonic approximation of the VTM for the cubic model is very good close to the threshold, where the pressure signal is almost sinusoidal, and quite good as γ approaches 0.4. The fact that the VTM cubic is better than the HBM cubic is a result of approximations having opposing effects. We also see that the three curves become one at the oscillation threshold.

Fig. 5 shows how P_1 varies with η and ζ. Firstly, the oscillation threshold γ_{th} decreases when ζ increases, whereas it increases when η increases. Indeed, η increases when the player increases the length of the pipe by closing tone holes. This therefore makes P_1 decrease, and the threshold of oscillation γ_{th} increase. This means that the player would have to blow harder to excite oscillation when the pipe becomes longer, at least for purely cylindrical pipes. For a real clarinet, however, the elongation of

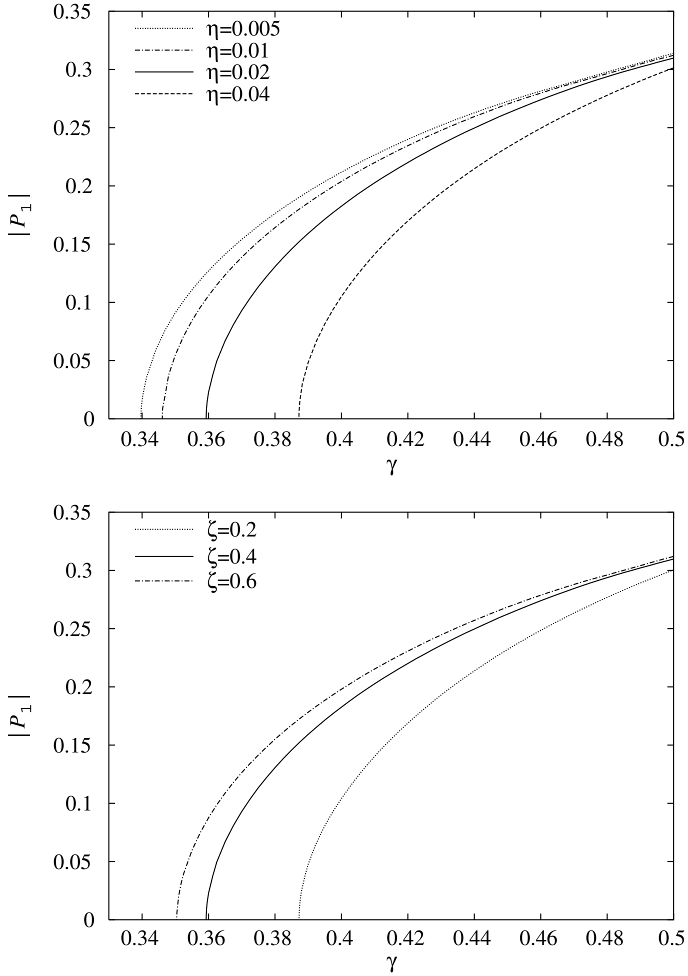

Fig. 5. Top: P_1 versus γ for different values of η, $\zeta = 0.4$ and $N_{\mathrm{p}} = 9$. Bottom: P_1 versus γ for different values of ζ, $\eta = 0.02$ and $N_{\mathrm{p}} = 9$.

the pipe is achieved by closing tone holes, and Fuks and Sundberg [24] have shown that the musician does not necessarily need to blow harder for notes involving a long part of the pipe.

It should be mentioned that similar figures to these have been published in [4], but with the continuation facilities of *Harmbal*, the results of the HBM may be followed quasi-continuously.

4.5. Register change

It is possible for a clarinet player to change the register and play the musical twelfth, which corresponds to the second eigenfrequency of the clarinet or the third harmonic, as the clarinet behaves as a closed-open pipe. This regime can be found using the HBM by setting as an initial condition a playing frequency equal to that of the third harmonic. As the impedance of this harmonic is smaller than that of the fundamental, the threshold, which now corresponds to $A = Y_3$., is greater.

From Fig. 6, the lowest register has higher acoustic pressure than the higher register for a given mouth pressure. At first this seems to conflict with the measurements of Fuks and Sundberg [24] and with the reports of clarinettists that the blowing pressure for a given dynamic level is largely independent of register. However, the sensitivity of the ear increases with increasing frequency up to about 3 kHz, so lower acoustic pressure should be necessary for the higher register to give the same perceived loudness. There is thus not necessarily a contradiction in this. Furthermore, our model does not take into account that in real performances, the reed may often operate in a beating regime, and the player may assist the upper register using the vocal tract. However, Fig. 6 shows that the threshold pressure increases for higher registers, which is in agreement with informal reports from clarinettists.

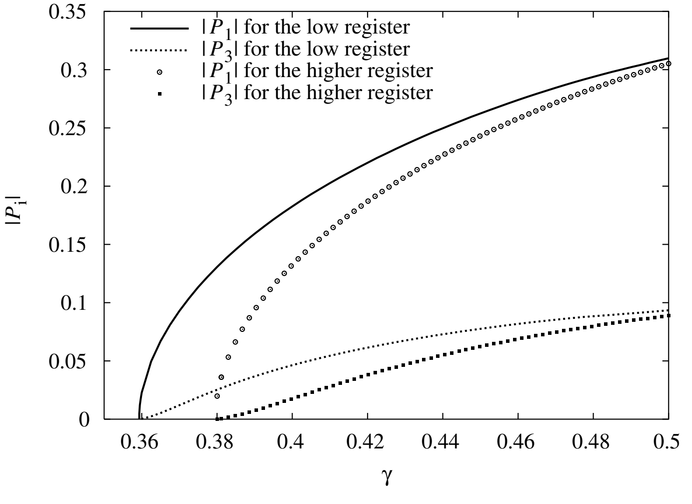

Fig. 6. Register of the fundamental tone of the clarinet and the register of the twelfth ($\zeta = 0.4$, $\eta = 0.02$, $N_p = 9$).

5. Effect of dispersion

Dispersion is the effect that the sound velocity varies with the frequency of the travelling wave. This results in inharmonicity of the pipe impedance, i.e., that the resonance frequencies of the pipe are no longer harmonically related.

At the oscillation threshold, the playing frequency f is still equal to the first pipe resonance f_p, but the latter is no longer equal to f_0. Instead, its value is determined by making the real part of Eq. (8) equal to π/2 so that

$$\frac{f_p}{f_0} = 1 - \frac{2}{\pi}\psi\eta\sqrt{\frac{f_p}{f_0}}. \tag{20}$$

When the oscillation amplitude grows, higher harmonics start to appear. The higher pipe resonances are shifted upwards (relative to the first one) due to dispersion, while the reed movement must stay periodic, i.e., harmonic. To maximize the energy, the playing frequency therefore increases with increasing importance of the higher harmonics, as the HBM shows in Fig. 7. Note that the first order of the HBM does only involve the first harmonic and is thus not able to capture the increase of the playing frequency with increasing γ. The change from the threshold to $\gamma = 0.5$ is above 0.8% (about 14 cents). The difference limen for a perceptible pitch shift of similar sounds is around 0.2% (about 4 cents) [25]. Note that the clarinet player compensates for this effect by altering the embouchure.

The amplitude of the different harmonics do not change significantly when dispersion is added to the model, as seen in Fig. 8. However, the relative phase of the harmonics changes with γ for the same reason as the playing frequency changes: higher, inharmonic partials of the pipe become important and shift the phases of the harmonics of the reed movement.

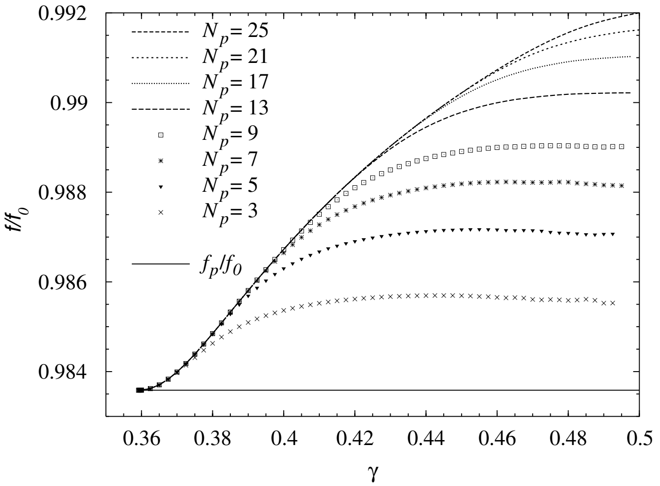

Fig. 7. Playing frequency using the HBM for different N_p for $f_0 = 100$ Hz and dispersion ($\zeta = 0.4$, $\eta = 0.02$). The line f_p/f_0, given by Eq. (20), indicates the fundamental of the pipe.

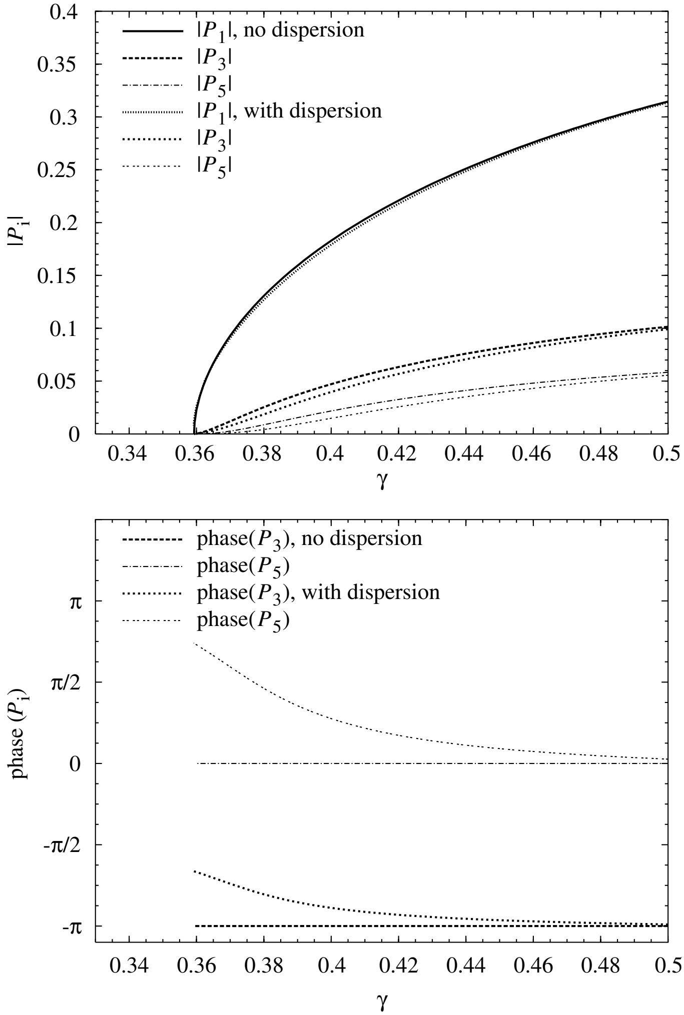

Fig. 8. Spectrum with and without dispersion (HBM, $N_p = 25$, $\zeta = 0.4$, $\eta = 0.02$). Top: modulus. Bottom: phase (P_1 real).

6. Influence of the reed resonance

In order to study the different effects separately, we now ignore dispersion but treat the reed as a spring with mass and damping using Eq. (4). From now on this will be referred to as Model B. (Model A is the simple reed model). Common values for the reed characteristics are [6]: $\omega_r = 23250\ s^{-1}$ ($f_r = 3700$ Hz), $g_r = 2900\ s^{-1}$, and $\mu_r = 0.0231\ kg\ m^{-2}$.

6.1. Pipe resonances independent of reed resonance

As was the case when dispersion was included in the model, the playing frequency changes with the mouth pressure. In the study by Kergomard and Gilbert [26] of some aspects of the role of the reed, an approximation of the frequency is given. In the case where the first resonance of the pipe is far below the reed resonance:

$$\frac{f - f_\mathrm{p}}{f_\mathrm{p}} = -\frac{2}{\pi\sqrt{3}} R\zeta \left[1 + \frac{3}{4}(\gamma - \gamma_t h)\right], \tag{21}$$

R having been defined in Section 2.1. The frequency at the threshold is thus given by $f_\mathrm{th} = (1 - (2\zeta R/\pi\sqrt{3}))$. The mouth pressure at the threshold is:

$$\gamma_\mathrm{th} \simeq \frac{1-\alpha^2}{3-\alpha^2} + 2\left(\frac{1-\alpha^2}{3-\alpha^2}\right)^{\frac{3}{2}} \frac{\psi\eta\sqrt{\alpha(f_\mathrm{r}/f_\mathrm{p})}}{\zeta} \tag{22}$$

where $\alpha = f_\mathrm{th}/f_\mathrm{r}$.

It is important to note that it is the damping of the reed, and not its mass, that makes the playing frequency change. Another point is that the damping of the reed decreases the playing frequency compared to the first frequency of the pipe (this effect was characterized by a length correction of the pipe by Nederveen [27]). However, the variation of the playing frequency as γ increases is different. The frequency decreases due to the damping of the reed whereas it increases when dispersion is taken into account. We will compare the effects in a numerical example.

Suppose the first frequency of the clarinet is 100 Hz ($\omega_p = 628.3\ \mathrm{s}^{-1}$), which implies that the effective reed resonance frequency is at the 37th harmonic of the pipe. In this case, the values of M and R are, respectively, 7.3×10^{-4} and 3.4×10^{-3}.

Fig. 9 shows that the playing frequency is slightly decreased (0.05%, i.e., not perceptible) compared to the simple reed approximation and that it varies little between the oscillation and beating thresholds, about 100 times less than the variation caused by dispersion (cf Fig. 7, note the differing axes). Thus, as far as the playing frequency is concerned, the effect of the reed resonance is negligibly small when the pipe resonance is far from the reed resonance. It is also interesting to see that Eq. (21) is a good approximation in the entire regime.

The change in the spectrum is also quite small, as shown in Fig. 10. The phase difference is completely negligible, and the magnitudes differ by less than 1%.

The approximation of the reed as a simple spring is therefore rather good when the playing frequency is much smaller than the reed resonance frequency.

6.2. Pipe resonance interacting with reed resonance

Consider now a hypothetical clarinet with a pipe seven times shorter such that the first resonance frequency of the pipe is $f_\mathrm{p} = 700$ Hz. The fifth harmonic is thus just below the reed resonance frequency. In this case, the values of M and R are, respectively, 3.6×10^{-2} and 2.4×10^{-2}. Fig. 11 shows one period of the corresponding solution

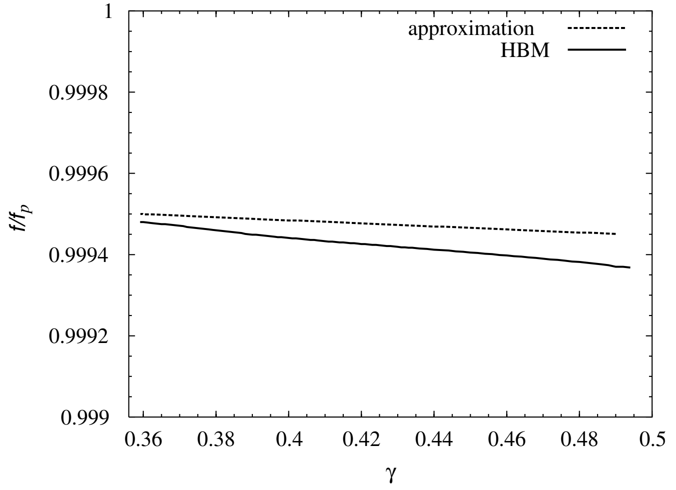

Fig. 9. Variation of the playing frequency with the mouth pressure when the mass and damping of the reed are taken into account: comparison between HBM and approximation (21) for $f_p = 100$ Hz, $N_p = 9$, $\zeta = 0.4$, $\eta = 0.02$.

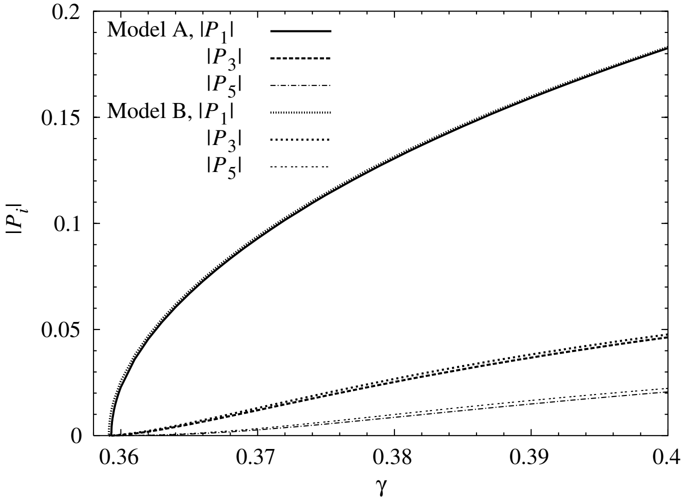

Fig. 10. Modulus of the harmonics for models A and B in the case $f_p = 100$ Hz ($\zeta = 0.4$, $\eta = 0.02$, $N_p = 9$). γ is cut at 0.4 as the reed effect is largest near the oscillation threshold.

(model B) as well as a period for $M = R = 0$ (model A). The solution for model B was obtained from the solution for model A by increasing M and R progressively.

The difference between models A and B as γ changes is shown in Fig. 12. Apart from the deviation of the harmonics, it is striking that the oscillation threshold is

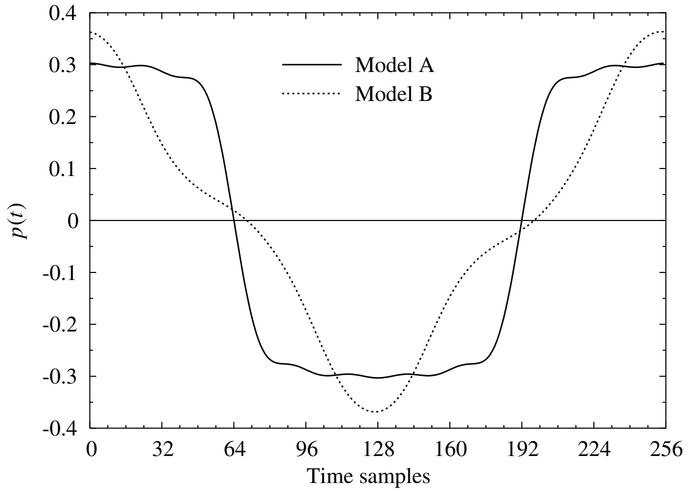

Fig. 11. Waveform $p(t)$ for models A and B when f_p = 700 Hz ($\gamma = 0.4$, $\eta = 0.02$, $\zeta = 0.4$, $N_p = 9$).

lowered so that a lower blowing pressure is needed to obtain a sound when the reed has mass and damping. This is explained by the fact that the third pipe resonance (the fifth harmonic) interacts with the peak resonance of the reed, and this stabilizes the oscillation [8].

The frequency changes by about 0.2% in the range of γ between the two thresholds, as shown in Fig. 13. This might not be perceptible by the human ear. but the deviation from model A is about 0.5%, which is above the difference limen [25]. Observe as well that approximation (21) is effectively no longer valid in the case where f_p is not far from f_r, especially for high mouth pressures.

7. Other regimes

The program can find many solutions but cannot decide about their stability. The clarinet produces a signal close to a square wave in the mouthpiece, and this is the solution that seems to be the most robust when changing one of the parameters. Other solutions seem to disappear or turn to the square-wave solution. This can happen when we change the number of harmonics N_p or when for instance γ is changed, and the evolution may depend on the way the change is made.

For example the two following solutions are considered: the first one is the square solution and will be called solution A in the following. The other ones, called solutions B and C, are nonsquare and have the shapes as shown in Fig. 14. The first resonance of the pipe is 100 Hz, so the reed may be considered as a simple spring. But dispersion is taken into account.

In Figs. 15 and 16, the different components of the pressure are represented as a function of γ for these solutions. They were obtained by decreasing γ. We can

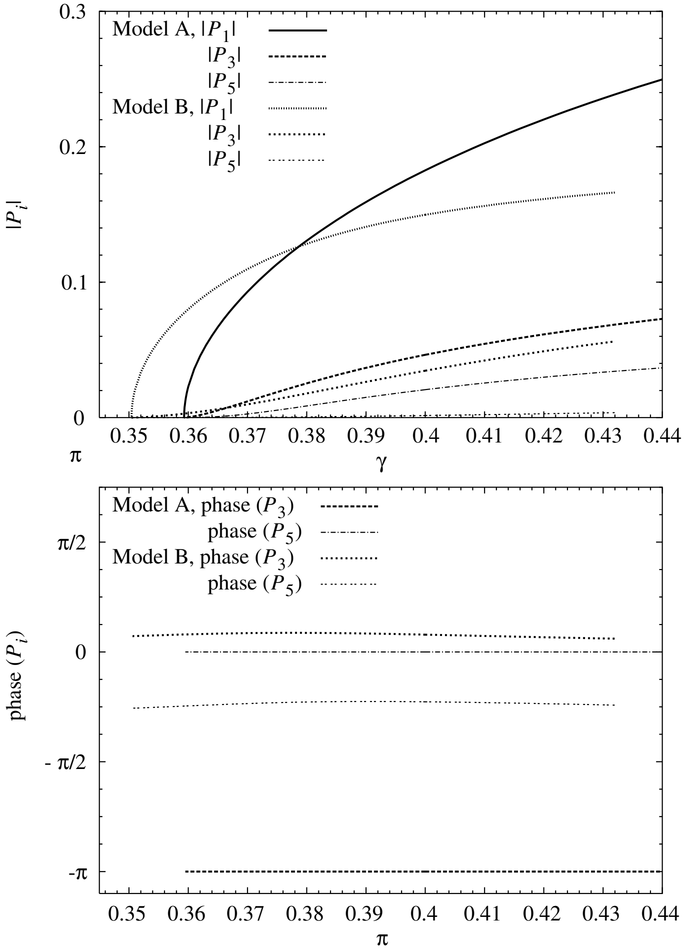

Fig. 12. The harmonics for models A and B, for f_p = 700 Hz. Top: modulus. Bottom: phase (P_1 chosen real).

observe a jump for solutions B and C. However, if γ is increased from the threshold, only the square solution is obtained.

The variation of the playing frequency versus γ can as well give further details about these solutions. Solutions B and C seem indeed to operate at another frequency than solution A, as shown in Fig. 17.

These different solutions correspond to cases where the first harmonic is not necessarily the largest one. The third one can indeed be much larger, like for solution C, or even the seventh or the ninth one, as for solution B. These "other" solutions appear only for large γ because it is only above $\gamma = 0.42$ that the higher harmonics have a non-negligible influence, as shown in Fig. 2.

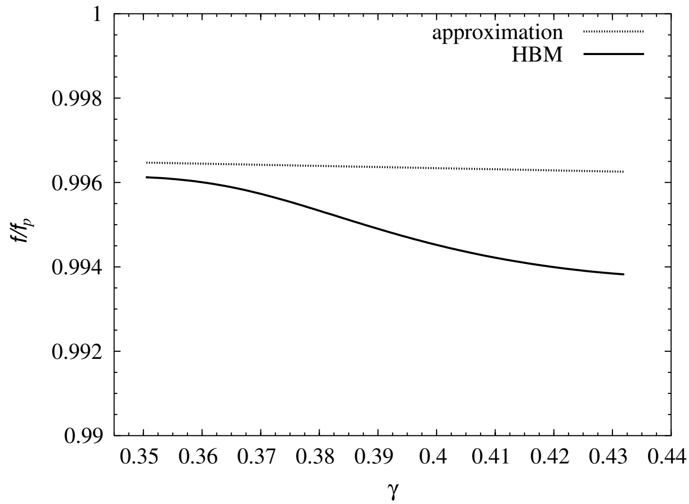

Fig. 13. Variation of the frequency with the mouth pressure when the mass and damping of the reed are taken into account, in the case of f_p = 700 Hz. Comparison to the approximation (21) a priori only valid if f_p is far from f_r.

Another feature of these solutions is that they cannot necessarily be retrieved for higher values of N_p. Thus solution C is not found above 13 harmonics and solution B above 29 harmonics. However, with other values of the parameters (for instance $\eta = 0.01$), they can be found for a hundred of harmonics.

So it is important to be aware that many solutions can be found with the HBM but some are not retrieved when N_p changes because the beating regime threshold changes with N_p (and the solution is therefore in a domain where the model is not accurate), or when γ changes slightly. A question that then arises concerns the physicality of such solutions.

8. Comparison with real clarinet

It is interesting to compare the spectra of a clarinet played by a real musician with that of the model, in spite of the simplifications of the latter. Because the model predicts several different possible solutions for different oscillation modes with different spectra, a clarinettist was asked to attempt to produce notes with unusual spectral envelopes, using a spectrum analyzer as visual feedback. Such notes are difficult to produce on a clarinet played normally, and require that the clarinettist modify his embouchure.

Fig. 18 shows two spectra produced using the same fingering but different embouchures. The first is that for F3 ($f \simeq 156$ Hz, written G3). With this fingering, all holes are closed except for the three most remote from the mouthpiece. The F3 is played normally and the sound recorded on the instrument axis at the end of the bell. This

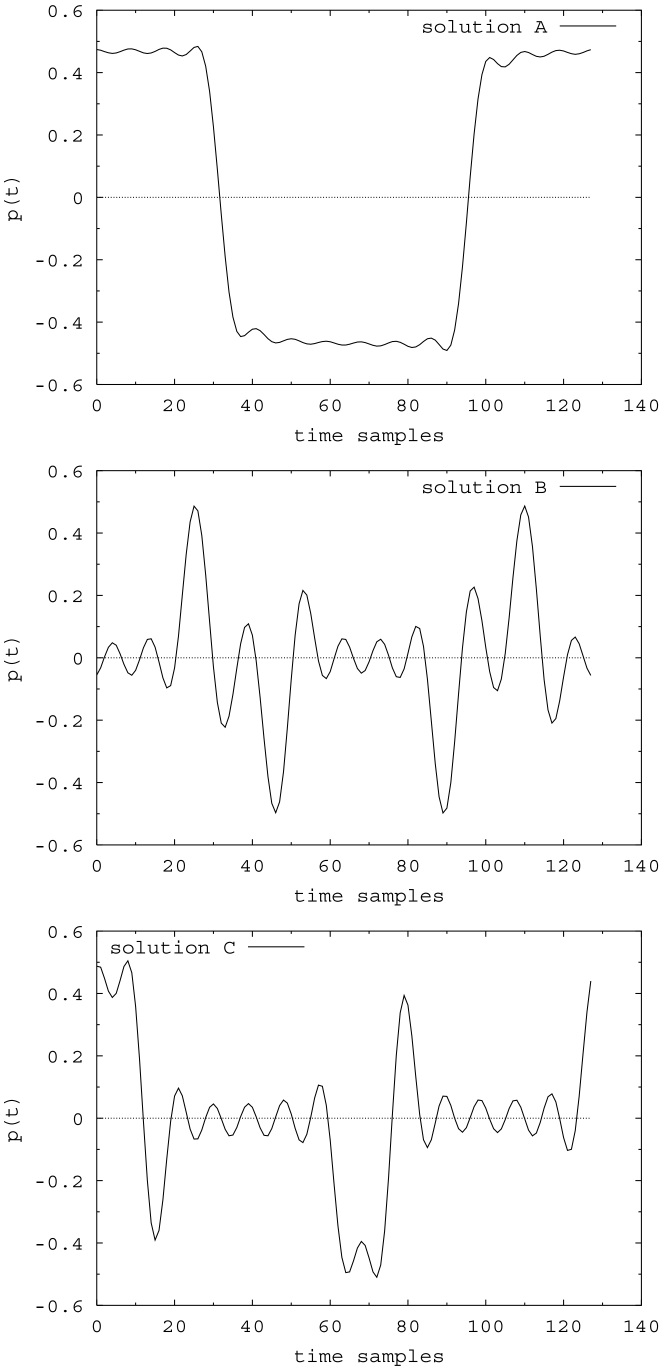

Fig. 14. One period of three different pressure waves $p(t)$ for the same set of parameters: $N_p = 13$, $\gamma = 0.485$, $\zeta = 0.4$ and $\eta = 0.02$.

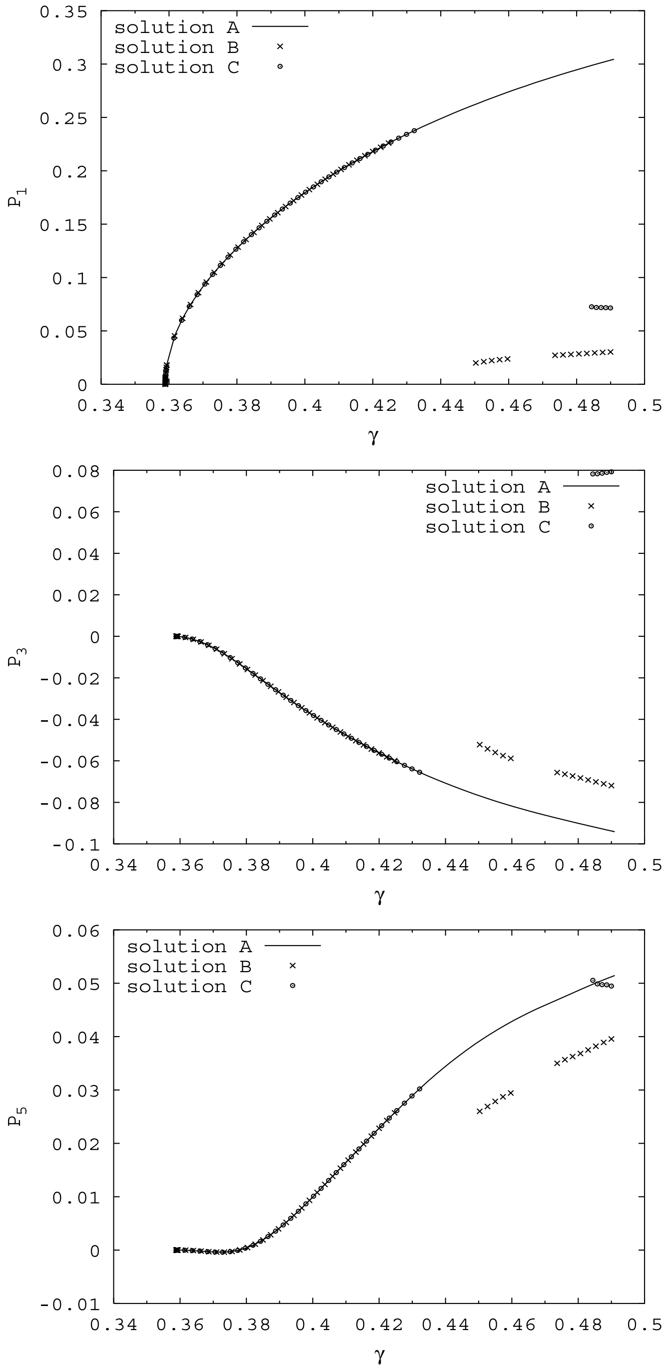

Fig. 15. P_i ($i = 1,\ldots,5$) versus γ for solutions A, B and C with $N_p = 13$, $\zeta = 0.4$ and $\eta = 0.02$.

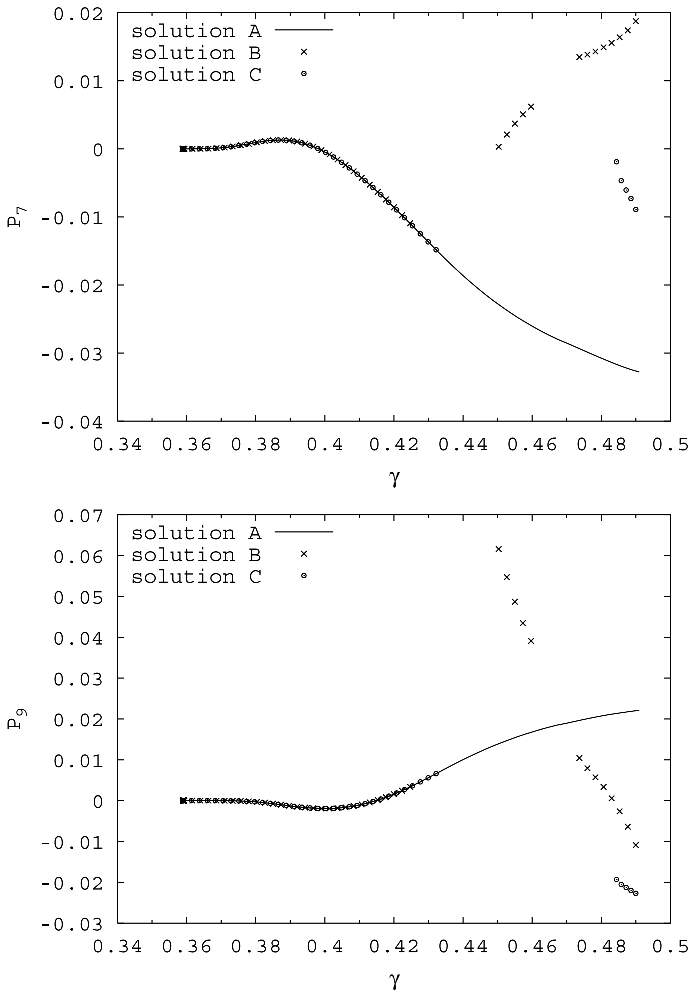

Fig. 16. P_i ($i = 7,9$) versus γ for solutions A, B and C with $N_p = 13$, $\zeta = 0.4$ and $\eta = 0.02$.

fingering can also readily be used to play a note in the next register, C5 ($f \simeq 523$ Hz), by altering the embouchure rather than by opening the register hole (spectrum not shown). Also with this fingering, it is possible to play F3 with a third harmonic that is so strong that it can be heard simultaneously as a separate note. The second spectrum of Fig. 18 shows this. This sounds like a chord made from a weak F3 and a stronger C5. However, this playing regime is difficult for the player to sustain: it tends to "jump" either to F3 or C5.

Note that in the second case, although the third harmonic is easily the strongest, the fundamental of F3 is still present and that only the odd harmonics of F3 are present. Although the strongest spectral component is that for C5, the corresponding

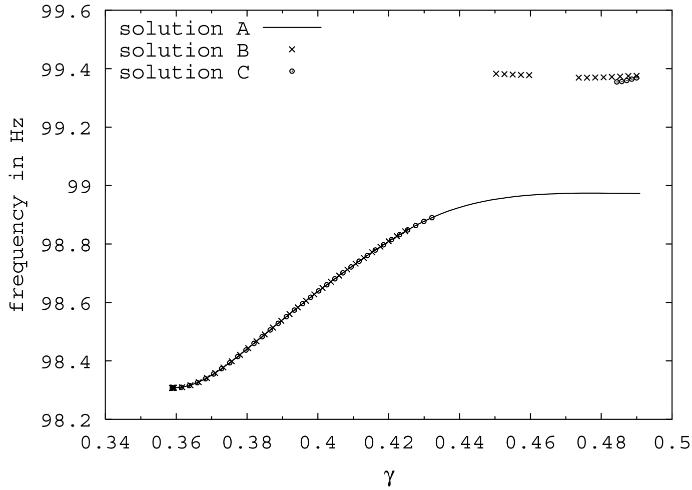

Fig. 17. Playing frequency for solutions A, B and C, as a function of γ.

note in the second register, this is quite different from a note played in the second register, especially when the speaker key is used as a register hole. For such notes, there is no measurable power in frequencies corresponding to the first register, and the even harmonics are not in general much weaker than the odd harmonics. (A database of clarinet sound spectra and impedance spectra is at [18].)

So clarinettists can produce very different spectral envelopes. To do so, however, they use modifications in several of the embouchure parameters, and perhaps the vocal tract. So this ability is not comparable with the ability of the HBM to find various solutions for the same sets of values of the parameters.

9. Conclusion

In the simplest model, which assumes no dispersion and the reed to be a simple spring without mass and damping, we found good agreement between the variable truncation method (VTM) and the harmonic balance method (HBM) close to the oscillation threshold. The playing frequency was equal to the first resonance of the pipe, and the Fourier components of the pressure in the mouthpiece were real.

By adding dispersion to the model, the playing frequency was significantly lowered (1–2%), especially close to the oscillation threshold (small γ). Here, the harmonics of the pressure showed large deviation from the nondispersive case, the phase in particular.

When common values for the mass, damping, and stiffness of the reed were introduced (but no dispersion), there was only a minor change for ω_r far from ω_p, while for $\omega_r/\omega_p \simeq 5.3$, there was a small lowering of the playing frequency (of about 0.5%), a significant one for the oscillation threshold, and a phase shift of the higher

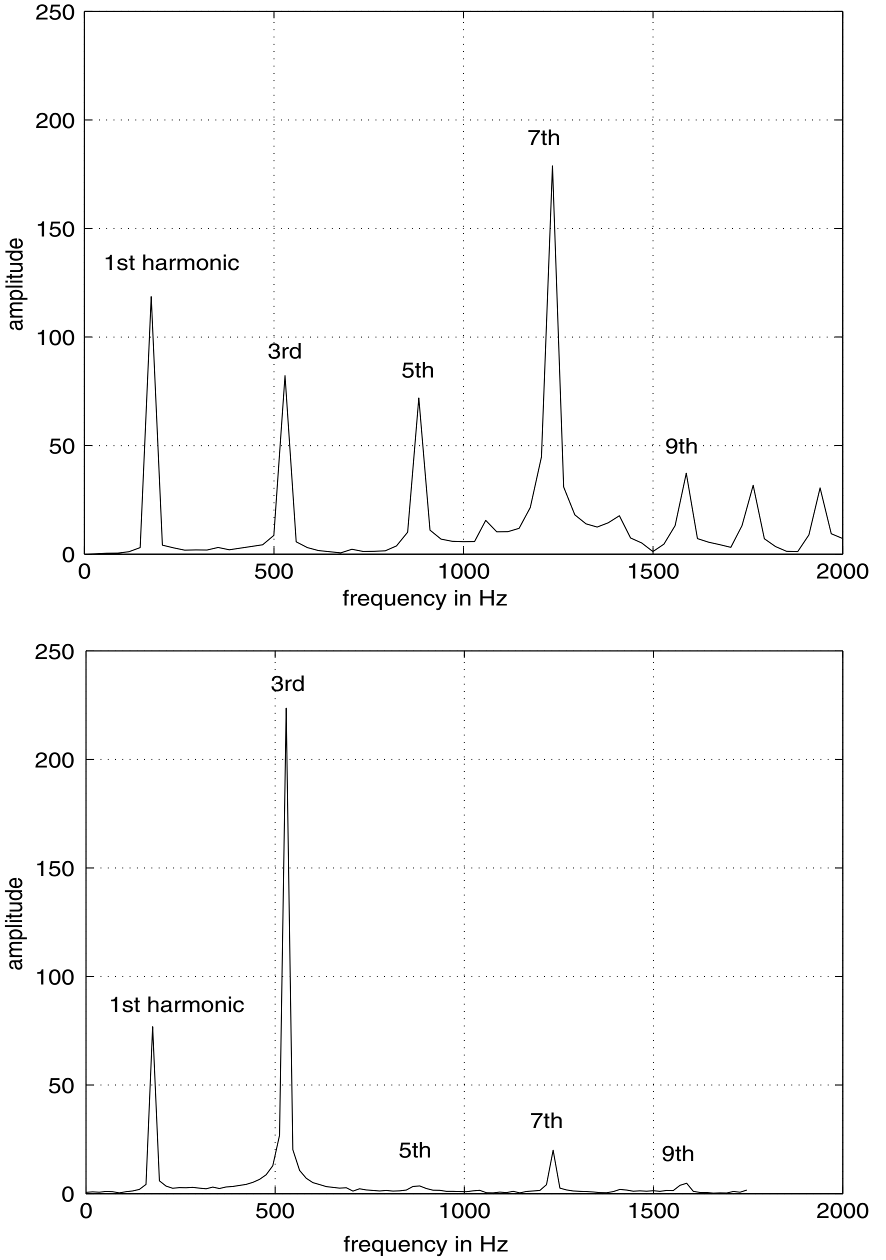

Fig. 18. Spectra of different sounds recorded at the end of the clarinetbore. Top: note F3. Bottom: player sounding both F3 and C5 together.

harmonics of the pressure in the mouthpiece. Mass and damping should thus be considered when the first resonance of the pipe is quite close to the reed resonance.

In our study we have encountered problems related to the physical stability of the solutions found by the HBM. When the number of harmonics N_p is increased, more solutions can be found, but it seems that only the rounded square solutions are maintained when N_p is increased (above 13 and 29 in our two cases). A study of the stability of the solutions is required.

The program *Harmbal* offers the possibility of detailed studies of the control parameters, but also for example the influence of the vocal tract or the compliance of the reed and the reed flow. A model of a real clarinet for any note could then be obtained by using a database of clarinet impedance measurements, available for example at [18], instead of an approximate analytical function.

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